CONTRADICTION WITHIN PURE NUMBER THEORY BECAUSE OF A SYSTEM-INTERNAL 'CONSISTENCY'-DEDUCTION

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I declared the refutability of proving mathematics on actual infinitudes as well as on a potential infinity at Leeds in 1967 (August 21); the JSL-editor who had invited me did not agree, afterwards, with "..paradoxical consistency.." in my intended title, because a summer school could not be responsible for the stop of "unzählige Forschungsvorhaben".[1] As to the inconsistency of classical analysis and of pure number theory, I refer to my papers [1] [2] [3] [5] and [1] [2] [3] [4] [6] [7] respectively.

Unsolved problems, like the hypothesis of Riemann $(\zeta(\sigma+it)=0 \wedge \wedge\, t \neq 0 \rightarrow \sigma=\frac{1}{2})$ and the "simpler" conjectures: of Cayley 1879 (4-color-problem), Goldbach 1742 $(n>1 \rightarrow V_{p_1} V_{p_2}\, 2n = p_1 + p_2)$ Fermat $(a^n + b^n = c^n \rightarrow n = 2 \vee n = 1)$ or the antique conjecture on perfect numbers $(\sum_{d|n} d = 2n \rightarrow 2|n)$, have no sense within formalized theories of the "real" continuum and of "natural" numbers respectively, as soon as untruths like 1=0 are formally deduced therein; such problems get back an artificial sense only, if you call for finite limits concerning the length of proofs or of definitions or of substitutes that stand for a variable as it is necessary in order to save the formal consistency of "general" principles. The justification of mathematizing is an arsenal of self-limiting algorithms (escorted by geometric imagination) for reasonable applications – rather than a collection of "eternal" truths, since the theorems end up with nonsense according to a formal control of our postulates in the most "exact" science.

0.1 I shall summarize my second private course of lectures at Bonn (16 sessions between January 31 and June 22, 1973; averaging > 150 min/session; 5 students). The refutation of arithmetic has to explain why Gentzen's consistency-"proof" 1936 by transfinite induction up to $\varepsilon_0 = \ldots^{\omega^{\omega}}$ can be undetermined by help of a system-internal deduction for the consistency-critical proposition which arithmetizes the system's consistency – the mathematicians of today think that it must be "undecidable" within the system.

Bernays claims in his critique of [1] that Gödel's proof 1931 for the underivability of that 'consistency'-proposition is no indirect proof. [2]

I insist on my statement that a demonstration for a semantical assertion with the prefix un- and the suffix -able is indirect, even if "derivation"/"derivability" has a finite/finitary sense; the only exception would be a finitary induction (on derivations) for a semiotic unequality form.[3]

The mathematician, who introduces a hypothetical assumption in order to negate it afterwards by reductio ad absurdum, reflects on a thing, that is nowhere available, as if it were exhibited directly – he or she is in a day-dream, beginning with that assumption, but they do not awake from the contradiction and do confound the event in their internal world with a contradictory statement of eternal validity about their external world. (The deduction theorem eliminates the inference "if $A \vdash B$ then $\vdash A \rightarrow B$"; formula (23) on p. 196 in K. Gödel 1931 is what can be derived thus. The metamathematical assumption $\vdash A$, however, cannot be direct, until a way of

how to derive A is shown; and the metamathematical assumption $\vdash \mathbf{A}$ remains indirect, if $\nvdash \mathbf{A}$ is the result to be obtained: [1] p. 303, lines 5-12.)

Moreover, Gödel 1931 as well as Hilbert & Bernays 1939/1970 consider two possibilities only, presumably since of a trust in the "law" of excluded middle: a formal system is either consistent or inconsistent; therefore (apparently), either the 'consistency'-proposition is underivable within the system or every proposition is derivable therein.

I realize a third possibility: prove the inconsistency first and foremost by way of a system-internal 'consistency'-deduction which is not only direct but also finite (within the power of man)!

0.2 In the case of classical analysis it is clear, according to [5] , that all solving sets for the transfinitely typized extension of Gödel's Dialectica re-interpretation of logical particles, as concerns any deduction within the system $\overline{\mathfrak{S}_1}$, are elements of the constructible set

$$\bigcup_{m} \{\sim \phi_m\} \cup \bigcup_{m_1} \left\{ \bigcup_{\substack{m_2 \\ p^c m_2 = p^c m_1}} \{\sim \phi_{m_2}\} \right\}$$

the order $\mathring{o}$ $(< \omega_1)$ of which is a majorant of Λ (: see [5] 4.), since all solving sets have orders $< \mathring{o}$ (: see Gödel 1940, 9.5); consequently there exists a finitary re-interpretation of $\overline{\mathfrak{S}_1}$ within a proper subsystem of $\overline{\mathfrak{S}_1}$ over ordinal numbers up to an $\overline{\mathfrak{S}_1}$ accessible limit.

0.3 In the graver case of pure number theory I utilize the profound and comprehensive (although undefined) idea of "Aufgabe", i.e. problem /task/exercise, owing to A.N. Kolmogorov 1925: it effects an admirable interpretation for the implication of intuitionistic logic and arithmetic, which can be extended to an elegant translation of the classical implication; Lorenzen 1955/1969 missed a revolutionary result, since he defines and symbolizes the interpretation of constructive implications by "operative" productions on different levels unfortunately in an informal manner, which may be "constructive", but knowingly evades any formal examination; my total formalization [4] [6] of that secret shuttle-service between $\rightarrow, \dot{\rightarrow}, \ddot{\rightarrow}, \ldots$ as imperative signals and $\rightarrow, \dot{\rightarrow}, \ldots$ meaning-free signs has disclosed the practicability of an "elementary" consistency-proof for pure arithmetic, which is finitary -and (therefore) direct -, i.e. it uses semiotic induction for such propositional forms only, which contain no constructive reflection; thereupon an inconsistency-deduction within formalized Peano arithmetic is inevitable, because the arithmetized counterpart of that finitary consistency-proof leads to a system-internal induction on a formal number-variable for a special predicate form which turns out to be primitive recursive [7], and which has an arithmetical description, though now it necessitates formally a fixed number of (constructively reflecting) generalized implications.

1. I try to outline my elaborate tool $\mathfrak{K}_0$ its latest execution consists of $145 = (2+2+3+10+30+8+13)+(28+2+12)+((13+2)+4+2)+14$ creative rules on 5 pages; the rules prescribe a finitary definition of $7+3+4+1$ relators, to which 1, 2, 3, 4 words are attached respectively. Such words are built up from 5 constants $\circ\ 0\ \xi\ \vee\ \wedge$ by help of $2+8+3+1$ functors $'\ \backslash\ \cup\ =\ \rightarrow\ \wedge\ \vee\ \vee\ \wedge\ \Rightarrow\ ,\ +\ \times\ <\ ,\ \vdash\ ,$ with which 1, 2, 3, 4 words are connected respectively;[4] the odd numbers 1, 3, .., 35, 37 will arithmetize successively these $5+14$ signs, and the Gödel number of ${}^f w_1 \ldots w_n$ will be $G({}^f w_1 \ldots w_n) = 2^{G({}^f)} \cdot 3^{G(w_1)} \cdot \ldots \cdot p_n^{G(w_n)}$ where $n \leq 4$, $p_4 = 11$.

The canonical system $\mathfrak{K}_0$ is a conservative extension of its subcalculus $\mathfrak{B}_0$ which formalizes pure number theory in a peculiar design and consists of $75 = (2+2+3+10+30)+17+0+11$ creative rules on 2 pages.

The constitution of $\mathfrak{K}_0$ may sink into oblivion, as soon as the arithmetization of those 15 relators $\mathfrak{q}$ by means of primitive recursive predicates $R_{\mathfrak{q}}$ is performed; since of uniqueness properties and of primitive recursive computability we can replace 1 2-place $R_{\mathfrak{q}}$ by a 1-place predicate, 1 4-place $R_{\mathfrak{q}}$ by a 3-place function $r_{\mathfrak{q}}$, 1 2-place $R_{\mathfrak{q}}$ by a 1-place $r_{\mathfrak{q}}$, 4 3-place $R_{\mathfrak{q}}$'s by 2-place $r_{\mathfrak{q}}$'s, and 1 2-place $r_{\mathfrak{q}}$ by a 1-place function. Without knowledge of the intended meaning that causes $\mathfrak{K}_0$, a study of the $R_{\mathfrak{q}}$'s or its derivates will surely be a puzzle.

1.1 $\mathfrak{K}_0$ aims at a 1-dimensional notation of *'derivations'* (by help of the 'derivator' $\vdash$), wherein all deriving steps are assembled, so that each essential detail of a derivation occurs in in its notation; after that a detail-number can be computed in the derivation-number. I sketch the derivation process on using an inofficial (2-dimensional) notation:

$$
{}^{=}00,\ \frac{{}^{=}kl}{{}^{=}{}'k'l},\ {}^{+}000,\ \frac{{}^{+}jol}{{}^{+}{}'jo'l},\ \frac{{}^{+}jkl}{{}^{+}j'k'l},\ {}^{\times}000,\ \frac{{}^{\times}jol}{{}^{\times}{}'jol},\ \frac{{}^{\times}jkh\quad {}^{+}hjl}{{}^{\times}j'kl};
$$

$$
\frac{U\quad V}{{}^{\wedge}UV},\ \frac{U}{{}^{\vee}UV},\ \frac{U}{{}^{\vee}VU},\ \frac{U(k_0)}{{}^{V}x\,U(x)},\ \frac{U}{{}^{\rightarrow}\cdot U},\ \frac{{}^{\rightarrow}AB\quad U_A(k_1..)}{V_B(k_1..)},\ \frac{{}^{\rightarrow}\cdot U}{U},\ \frac{V\quad {}^{\rightarrow}VU}{U},
$$

$$
\frac{{}^{\wedge}x\,U(x)}{U(k)}.
$$

(${}^{\rightarrow}AB$ will have to be $\mathfrak{Z}_0$-deduced, U_A, V_B will be the imperative translations of A, B on their lowest joint production-level; K_1 .. denote constant substitutes 0, $'0$, $''0$, for all free variables in ${}^{\rightarrow}AB$ (and in ${}^{\rightarrow}U_AV_B$). U, V translate two number-theoretic propositions (with no free variable) on their lowest joint level before/ after each deriving step.)[5]

V, $\wedge$ are translated as ${}^{=}00$, ${}^{=}{}'00$; all other prime formulas ${}^{=}t_1t_2$, ${}^{+}t_1t_2t_3$, ${}^{\times}t_1t_2t_3$ are translated into themselves – on level 0. ${}^{\rightarrow}\cdot U$ translates A on level $'l$ if U does it on level l; if level l_0 is the lowest joint level on which A,B have translations U and V, then ${}^{\rightarrow}UV$ is the translation of ${}^{\rightarrow}AB$ on level $'l_0$ as the lowest level; if level l_0 is the lowest level on which A(x) has a translation U(x), then ${}^{\wedge}x\,U(x)$ is the translation of ${}^{\wedge}x\,A(x)$ on level $'l_0$ as the lowest one. Similarly with ${}^{\wedge}$, ${}^{\vee}$ instead of ${}^{\rightarrow}$ and of ${}^{\rightarrow}$, or with V instead of $\wedge$, but within the same level – without an elevated level $'l_0$.[6]

1.2 Likewise $\mathfrak{Z}_0$ provides a 1-dimensional notation of *'deductions'* (by help of the 'deductor' $\angle$), wherein all deducting steps are itemized. I sketch the deduction process by use of an inofficial notation:

$$
{}^{\rightarrow}AA,\ \frac{{}^{\rightarrow}AC\quad {}^{\rightarrow}CB}{{}^{\rightarrow}AB},\ {}^{\rightarrow}{}^{\wedge}ABA,\ {}^{\rightarrow}{}^{\wedge}ABB,\ \frac{{}^{\rightarrow}CA\quad {}^{\rightarrow}CB}{{}^{\rightarrow}C{}^{\wedge}AB},\ {}^{\rightarrow}A{}^{\vee}AB,\ {}^{\rightarrow}B{}^{\vee}AB,\ \frac{{}^{\rightarrow}AC\quad {}^{\rightarrow}BC}{{}^{\rightarrow}{}^{\vee}ABC},
$$

$$
\frac{{}^{\rightarrow}{}^{\wedge}CAB}{{}^{\rightarrow}C{}^{\rightarrow}AB},\ \frac{{}^{\rightarrow}C{}^{\rightarrow}AB}{{}^{\rightarrow}{}^{\wedge}CAB},\ {}^{\rightarrow}AV,\ {}^{\rightarrow}\wedge A,\ {}^{\rightarrow}A(t)\,{}^{V}x\,A(x),\ \frac{{}^{\rightarrow}A(x)\,\dot{C}}{{}^{\rightarrow}{}^{V}x\,A(x)\,\dot{C}},\ {}^{\rightarrow}{}^{\wedge}x\,A(x)\,A(t),
$$

$$
\frac{{}^{\rightarrow}\dot{C}\,A(x)}{{}^{\rightarrow}\dot{C}\,{}^{\wedge}x\,A(x)};\ \frac{{}^{\rightarrow}A(x)\,A('x)}{{}^{\rightarrow}A(0)\,A(x)},\ {}^{\rightarrow}V{}^{=}00,\ {}^{\rightarrow}{}^{=}\xi\eta{}^{=}{}'\xi'\eta,\ {}^{\rightarrow}{}^{=}{}'\xi'\eta{}^{=}\xi\eta,\ {}^{\rightarrow}{}^{=}{}'\xi 0\wedge,\ {}^{\rightarrow}{}^{=}0'\xi\wedge,
$$

$$
{}^{\rightarrow}{}^{=}\xi\zeta{}^{+}\xi 0\zeta,\ {}^{\rightarrow}{}^{+}\xi 0\zeta{}^{=}\xi\zeta,\ {}^{\rightarrow}D^{+}(\xi\eta\zeta)\,{}^{+}\xi'\eta'\zeta,\ {}^{\rightarrow}{}^{+}\xi'\eta'\zeta\ D^{+}(\xi\eta\zeta),\ {}^{\rightarrow}{}^{=}\zeta 0{}^{\times}\xi 0\zeta,\ {}^{\rightarrow}{}^{\times}\xi 0\zeta{}^{=}\zeta 0
$$

$$
{}^{\rightarrow}D^{\times}(\xi\eta\zeta)\,{}^{\times}\xi'\eta\zeta,\ {}^{\rightarrow}{}^{\times}\xi'\eta\zeta\,D^{\times}(\xi\eta\zeta)\ \text{where}\ D^{+}(\xi\eta\zeta) \leftrightarrow {}^{V}\xi_0\,{}^{V}\eta_0\,{}^{V}\zeta_0\,{}^{\wedge\wedge\wedge}{}^{=}\xi\xi_0{}^{=}\eta\eta_0{}^{=}\zeta\zeta_0{}^{+}\xi_0\eta_0\zeta_0,
$$

$$
D^{\times}(\xi\eta\zeta) \leftrightarrow {}^{V}\xi_0\,{}^{V}\eta_0\,{}^{V}\zeta_0\,{}^{\wedge}{}^{=}\xi\xi_0{}^{\wedge}{}^{=}\eta\eta_0{}^{\wedge}{}^{+}\zeta_0\xi\zeta{}^{\times}\xi_0\eta_0\zeta_0
$$

are used as abbreviations. (The substitution of t for x in $A(x)$ is confusionless; $\dot{C}$ does not contain x free. x denotes variables ξ, $'\xi$, $''\xi$, ... alias ξ, ξ_1, ξ_2, ... ; t denotes terms k and x, $'x$, $''x$, ... $\xi\eta\zeta\,\xi_0\eta_0\zeta_0$ stand for $\xi\xi_1..\xi_5$.)

1.3 It is unusual to derive derivations, i.e. to "create" derivations instead of their results. The exhibition of deductions within $\mathfrak{Z}_0$ is lengthy too. (Nevertheless there will "exist" a $\mathfrak{Z}_0$-deduction

$$
\frac{{}^{\rightarrow}\ {}^{\uparrow}}{{}^{\rightarrow}V\wedge};
$$

its result is 'verum implies falsum'. A printing of such a deduction would obviously be "easier" in my official (1-dimensional) notation, whereas the tree-form provokes a commercially not available size for one leaf of paper.)

Let $\mathfrak{Z}$ be the formalism which erases each $\mathfrak{Z}_0$-deduction except for its result: ${}_{D}{}^{\angle}w_1w_2w_3 \longrightarrow {}_{S}w_3$, i.e. if '${}^{\angle}w_1w_2w_3$ is a deduction' has been noted down (in $\mathfrak{Z}_0$), then note down 'w_3

is a theorem/Satz' (in $\mathfrak{B}$). $\mathfrak{B}$ is equiconsistent with the usual formalizations of pure number theory: $\rightarrow \vee \mathbf{A} \rightarrow \mathbf{A} \wedge$, i.e., "more" inofficially, $\mathbf{A} \vee \neg \mathbf{A}$, is eliminable; 3.2 equality properties $\mathbf{x} = \mathbf{y} \rightarrow (\mathbf{A}(\mathbf{x}) \rightarrow \mathbf{A}(\mathbf{y}))$ as to the 3 places of $^+$and of $^\times$ are $\mathfrak{B}$-deducible; 2 uniqueness properties $\forall \mathbf{z}_0 \wedge \mathbf{z} (\mathbf{A}(\mathbf{z}) \leftrightarrow \mathbf{z}_0 = \mathbf{z})$ as to the 3rd place of $^+$ and of $^\times$ are $\mathfrak{B}$-deducible. The use of terms $\mathbf{x}+\mathbf{y}$ and $\mathbf{x} \cdot \mathbf{y}$, by which the current postulates for $+$, $\cdot$ are mixed up with $=$, can be expressed within $\mathfrak{B}$ by way of (eliminable)ι-description : $\mathbf{x}+\mathbf{y} = \iota_{\mathbf{z}} + \mathbf{xyz}$, $\mathbf{x} \cdot \mathbf{y} = \iota_{\mathbf{z}} \cdot \mathbf{xyz}$.

Officially each $\mathfrak{B}$-deduced theorem has the form $\rightarrow \mathbf{u}_1 \mathbf{u}_2$; e.g. $\rightarrow \forall \mathbf{u}$ for ' $\mathbf{u}$ is true'.

1.4 The "elementary" consistency-proof for $\mathfrak{B}$ *(0.3)* is based upon the fact that each $\mathfrak{K}_0$-derivation (: relator $_{\mathfrak{d}}$), whose result does not contain $\rightarrow, \wedge, \wedge, \vee, \forall$, can be transformed directly into a $\mathfrak{K}_0$-derivation without detour (for the same result), i.e. into a practical $\mathfrak{K}_0$-derivation (: relator $_{-\mathfrak{d}}$) as being created according to the 3 starts for $^=, ^+, ^\times$ and to the $1+2+2$ increasing "modi arithmeticorum" for $^=, ^+, ^\times$ (and to nothing else): see *1.1*

Indeed, hereafter each $\mathfrak{B}$-deduced equation without variables must have the form $\rightarrow \forall {}^=\mathbf{kk}$ since $\mathfrak{B}_0$ exhibits a deduction for each $\mathfrak{B}$-deduced formula $\rightarrow \forall {}^=\mathbf{kl}$ „ so that the derivation

$$\dfrac{\dfrac{*\quad\dagger}{\rightarrow \forall {}^=\mathbf{kl}} \quad {}^=00}{{}^=\mathbf{kl}}$$

can be transformed directly into a detour-free derivation

$$\dfrac{\dfrac{\dfrac{{}^=00}{{}^='0'0}}{\vdots}}{{}^=\mathbf{kk}}$$

. Therefore every deduction differs semiotically from $\dfrac{*\quad\dagger}{\rightarrow \forall \wedge}$ since of

$$\dfrac{\dfrac{*\quad\dagger}{\rightarrow \forall \wedge} \quad {}^=00}{{}^='00}$$

. The arithmetized image of that

consistency-proof takes the form: (*) $\vdash_{\mathfrak{B}} R_{\mathfrak{D}}(\xi) \wedge (\xi)_{3,1} = 7 \rightarrow (\xi)_{3,2} \neq 9$ where $(\mathbf{a})_{\mathbf{m}}$ denotes the exponent of $p_{\mathbf{m}}$ in the unique prime number representation of a, and where $(\mathbf{a})_{\mathbf{m},\mathbf{n}}$ abbreviates $((\mathbf{a})_{\mathbf{m}})_{\mathbf{n}}$; (*) follows from

$\vdash_{\mathfrak{B}} R_{-\mathfrak{d}}(\xi) \wedge (\xi)_{3,0} = 17 \rightarrow (\xi)_{3,1} = (\xi)_{3,2} \dot{\wedge} R_{-\mathfrak{d}}(\xi) \rightarrow (\xi)_3 \neq 2^{17} \cdot 3^{2^{11}} \cdot 3^3 \cdot 5^3 (:= G({}^='00))$,

as soon as the transformation-lemma will be $\mathfrak{B}$-deduced in an arithmetized version.

If you halt the deduction process at such a "desirable" stage of the plot, as viewed in the arithmetization mirror, then our illusion that proofs do prove something could be upheld. However, there is now the (comparatively short) deduction of $\vdash_{\mathfrak{B}} \ulcorner \nvdash_{\mathfrak{B}} \wedge \urcorner \rightarrow \wedge$ due to Kurt Gödel 1931.[7] Since (*) expresses $\vdash_{\mathfrak{B}} \ulcorner \nvdash_{\mathfrak{B}_0} .. \wedge \urcorner$, and since $\vdash_{\mathfrak{B}} \ulcorner \vdash_{\mathfrak{B}} \mathbf{A} \urcorner \rightarrow \ulcorner \vdash_{\mathfrak{B}_0} .. \mathbf{A} \urcorner$, hence $\vdash_{\mathfrak{B}} \ulcorner \nvdash_{\mathfrak{B}_0} .. \wedge \urcorner \rightarrow \ulcorner \nvdash_{\mathfrak{B}} \wedge \urcorner$, is easily verifiable, the consequence of (*) is $\vdash_{\mathfrak{B}} \wedge$, officially $_{\mathfrak{s}} \rightarrow \forall \wedge$. Henceforth we receive even $\vdash_{\mathfrak{K}_0} .. {}^='00$, as seen before and contrary to what had been "proved" already in the arithmetized version $\vdash_{\mathfrak{B}} \ulcorner \nvdash_{\mathfrak{K}_0} .. {}^='00$. (Let $..\mathbf{w}$ stand for a deduction/derivation whose result is $\mathbf{w}$.)

2. I discuss here 2 of the 23 (or 24) primitive recursive functions the composition of which executes the transformation of each derivation, a transformation as eliminating all subderivations that depend on number-theoretic deduction. It is clear that I select the eliminating functions for formal induction and for implication transitivity (chain inference), since Gentzen's valuation of deductive steps by ordinal numbers $< \epsilon_0$ ascends in such cases very quickly.

2.1 A derivative application of formal induction takes the form

$$\frac{\overset{\cdot}{A(0) \to A(x)} \qquad \overset{*}{\overset{\dagger}{U_A(k_1 \,..\, 0)}}}{U_A(k_1 \,..\, k)}$$

; let a be its number. If $k = 0$ (or if A does not contain x free), then $(a)_2$ has the same result as a has: $(a)_{2,3} = (a)_3$. If $k > 0$, then we have to apply k-fold the imperative translation of $A(x) \to A('x)$, i.e. the result of the subdeduction $(e)_2 = (a)_{1,2}$, in order to receive $U_A(k_1 \,..\, k \,..)$ independent of any use of the deduction $e = (a)_1$. (Consider that the tuple of free variables in $A(0) \to A(x)$ and the corresponding tuple for $A(x) \to A('x)$ can contain the induction variable x at different positions, since x does not occur in $A(0)$.)

Let $q(l, (a)_4)$ be a primitive recursive computation of the tuple numbers $\ulcorner k_1 \,..\, l \,..\urcorner$ in the tuple number $\ulcorner k_1 \,..\, k \urcorner = (a)_4$.[5] Let b be a varying derivation number and $R_D((b)_1)$, then there is a primitive recursive computation of $\ulcorner \to U_A V_B (k_1 \,..)\urcorner$ in the number $(b)_{1,3} = \ulcorner \to AB \urcorner$: $p((b)_{1,3}, (b)_4)$.

The transformed derivation $\tau_{6.17}(a)$, that eliminates the derivation a in the sense of $(\tau_{6.17}(a))_3 = (a)_3$ gets the number $\tau_{6.17}(a) = i(k_a, a)$, where k_a can be singled out from a by primitive recursive computation, so that k_a satisfies $(a)_4 = \ulcorner k_1 \,..\, k_a \urcorner$, and where i is defined by primitive recursion as follows:

$$i(0, a) = (a)_2$$

$$i('l, a) = 2^{37} \cdot 3^{(a)_{1,2}} \cdot 5^{i(l, a)} \cdot 7^{(p((a)_{1,2,3}, q(l, (a)_4)))_2} \cdot 11^{q(l, (a)_4)} .$$

2.2 A derivative application of implication transitivity takes the form

$$\frac{\dfrac{A \to C \quad C \to B}{A \to B} \qquad U_A(k_1 \,..)}{V_B(k_1 \,..)},$$

wherein the tuple of substitutes belongs to the tuple of free variables in $\to AB$; let a be the number of that derivation.

If e is a deduction number, then let $s((e)_3)$ be a primitive recursive computation for the lowest level of the imperative translations as to $(e)_3$. Now $s((a)_{1,3})$, $s((a)_{1,1,3})$, $s((a)_{1,2,3})$ may notify 3 different levels. Therefore I need an "elevator" for raising and lowering the level of the result of a derivation b:

$$s^+(b) = 2^{37} \cdot 3^b \cdot 5 \cdot 7^{2^{29} \cdot 3 \cdot 5^{(b)_3}} \cdot 11,$$

$$s^-(b) = 2^{37} \cdot 3 \cdot 5^b \cdot 7^{(b)_{3,2}} \cdot 11 \leftarrow (b)_3 = 2^{29} \cdot 3 \cdot 5^{(b)_{3,2}},$$

$$le(s_0, b) = \begin{cases} s^+(le(s_0 - 1, b)) & \\ b & \leftarrow s_0 \overset{>}{\underset{<}{=}} s_b = \text{level of } (b)_3 ; \\ s^-(le(s_0 + 1, b)) & \end{cases}$$

the defining condition of the third case leads to a primitive recursive expression. Both of s^+ and s^- lengthen a derivation; but $\breve{s}(b) = (b)_1 \leftarrow (b)_2 = 1$ shortens a lifted derivation b.

After such preparations the above mentioned application of $(a)_1 = e$ can be eliminated by applications of $(e)_1$ and of $(\bar{e})_2$, so that $(\tau_{6.2}(a))_3 = (a)_3$:

$$c_1(a) = 2^{37} \cdot 3^{(a)_{1,1}} \cdot 5^{le(s((a)_{1,1,3}) - 1, (a)_2)} \cdot 7^{(p((a)_{1,1,3}, q_1((a)_4)))_2} \cdot 11^{q_1((a)_4)}$$

$$c_2(c_1; a) = 2^{37} \cdot 3^{(a)_{1,2}} \cdot 5^{le(s((a)_{1,2,3}) - 1, c_1(a))} \cdot 7^{(p((a)_{1,2,3}, q_2((a)_4)))_2} \cdot 11^{q_2((a)_4)}$$

$$\tau_{6.2}(a) = le(s((a)_{1,3}) - 1, c_2(c_1; a)) .$$

q_i are primitive recursive functions which compute the tuple of substitutes for the tuple of free variables in $\begin{matrix} A \to C \\ C \to B \end{matrix}$ from the tuple number $(a)_4$, so that the substitutes in $q_i((a)_4)$ and in $(a)_4$

for the same variable (as contained free in $\begin{matrix} A \to C \\ C \to B \end{matrix}$ and in $A \to B$) are equal, whereas 0 is the substitute for free variables that occur in C and not in $A \to B$.

2.3 The eliminating functions for exportation and for generalisator-induction take the form $\tau_{\substack{6.9 \\ 6.16}}(a, d)$, wherein the parameter-derivation d satisfies $(d)_2 = le(s((a)_{1,3,2}), a)$: d applies that imperator/generalisator in the result $(a)_3$ of a, which corresponds to $(a)_{1,3,2,0}$ on the lowest level.

The finite list of parameters $d_1, \ldots, d_n$, which enter a composition of eliminating functions, is expressible uniformly in terms of one parameter $d = \prod_1^n{}_\nu\, p_{4+\nu}^{d_\nu}$.

2.4 The elimination of $\wedge \to A$ (ex falso quodlibet) and of $'\xi = 0 \to \wedge$, $0 = '\xi \to \wedge$ will use the fact that the result of each detour-free derivation differs from $'0 = 0$ (: see *1.4*) as well as from $'\xi = 0$ or $0 = '\xi$, i.e. the im perative translations are unapplicable, *if* an elimination of all preceding deductive results – they belong to shorter deductions – is already performed.

3. The total elimination of all number-theoretic results depends on a classification of deductions and of derivations.

3.1 A deduction e belongs to grading level l $(: gl_D^l(e))$, if e and all its subdeductions have results, the imperative translation of which has (lowest) levels $\leq l+1$, so that implicans and implicatum of those results are translated on levels $\leq l$. A derivation a belongs to grading level l $(: gl^l(a))$, if a and all its subderivations have results on levels $< l$, *and* if all deductions, that are applied within a, belong to grading level l (as defined before).

The first $8+4$ derivation modi under *1.1* are said to be independent in their last deriving step; they depend on $\bullet$ (emptiness). The last $2+3$ derivation modi under *1.1* are said to be dependent in their last deriving step, and that on the derivation/deduction of the premises U , $\to AB$, $\to \bullet U$, $\to VU$, $\wedge_x U(x)$ respectively.

A derivation a is said to be relatively independent with regard to grading level l $(: ri^l(a))$, if a and all its subderivations are independent and belonging to grading level l *or* depend on deductions/derivations which belong to grading levels $< l$.

3.2 Since the eliminating functions in a do not depend on a, but only on a few predecessors of a, there is a primitive recursive function $T(l, a, d)$ which executes all intermediate eliminations. So I receive:

$$gl_D^0((a)_1) \wedge ri^0((a)_2) \to ri^0(T(0, a)) \wedge (a)_3 = (T(0, a))_3 \ ,$$
$$gl^0(a) \to ri^0(T(0, a)) \wedge (a)_3 = (T(0, a))_3 \ ;$$
$$gl_D^{'l}((a)_1) \wedge ri^{'l}((a)_2) \wedge \widehat{ri}^l(d) \to ri^{'l}(T('l, a, d) \wedge (a)_3 = (T('l, a, d))_3 \ ,$$
$$gl^{'l}(a) \wedge \widehat{ri}^l(d) \to ri^{'l}(T('l, a, d) \wedge (a)_3 = (T('l, a, d))$$
$$\text{where } \widehat{ri}^l(b) \leftrightarrow \wedge\xi(\xi < b \to ri^l((b)_\xi)).$$

3.3 The induction on grading levels can be recapitulated as a formal induction on a formal number-variable for an induction predicate which is primitive recursive (and $\mathfrak{Z}$-expressible). Therefore the arithmetized transformation-lemma *(1.4)* has a finite $\mathfrak{Z}$-deduction – without any finitary chain of equations. The remark that ri^0 means "completely" independent (in all derivation steps) terminates the system-internal 'consistency'-deduction with respect to and within $\mathfrak{Z}$.

And it is well-known that $\mathfrak{Z}$ deduces a contradiction now. Quod erat demonstrandum.

NOTES

1. Cf. The Journal of Symbolic Logic 33 (1968), p. 490 lines 9/10, and 7.

2. Cf. Dialectica 25 (1971) 191 - 194 (and notes 8, 9 on p. 195), esp. pp. 192, 193.

3. Cf. [1] p. 303, lines 12 - 22.

4. [6] [7] are concerned with a 9-place $\vdash$, and with 4-place $\prec_{,}\rightarrow_{.}$

5. Substitutes like k_0, k_1 and $k_{1\cdot\cdot}$ are registered officially in each derivation step. The order of the numbers $k_{1\cdot\cdot}$ corresponds to the creation of a tuple of all variables (without repetition) that occur in a given formula as free variables.

7. $\ulcorner . \urcorner$ symbolizes arithmetization.

6. $\wedge^{+}000^{\times}000$ is translated on level 1 as $\rightarrow_{.}\wedge^{+}000^{\times}000$, but not as $\wedge\rightarrow_{.}{}^{+}000\rightarrow_{.}{}^{\times}000$.

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